0315-098

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Extended Essay

**Title: An Investigation of Arithmetic Operations**

## Abstract

This paper is an investigation of selected aspects of arithmetic operations, the first three of which are addition, multiplication, and exponentiation. The issue being explored is, first, to explain functions which generalize the three operations, and thus develop definitions of operations after exponentiation for natural numbers. Another aspect of the issue being explored is whether the following arithmetic operations can be defined on non-integral real numbers: two types of tetration (the operation after exponentiation), and one type of pentation (the operation after tetration). Also, some of the interesting properties of tetration are discussed.

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## Introduction

The three basic arithmetic operations, addition, multiplication, and exponentiation, are quite common in mathematics. In fact, it could be argued that they form the basis of our entire mathematical system. But what about operations "after" exponentiation in this sequence of operations? This essay will be concerned with generalizing arithmetic operations, and also with defining and analyzing the properties of operations after exponentiation.

## The Two Generalized Exponentials

Addition, multiplication, and exponentiation are, of course, closely related: multiplication is repeated addition, and exponentiation is repeated multiplication. It is possible to define two different recursive functions which incorporate these operations and generalize them. The first, the Ackermann generalized exponential, works as follows (it is named $f_1$ here because it is arbitrarily the first of the two possible functions in this essay):

$$f_1(0,x,y)=y+x,$$
$$f_1(1,x,y)=y\cdot x,$$
$$f_1(2,x,y)=y^x,$$
$$\vdots$$

Here is a more formal, recursive definition of the Ackermann function, where x, y, and z are natural numbers (including zero):

1. $f_1(0,0,y)=y,$
2. $f_1(0,x+1,y)=f_1(0,x,y)+1,$
3. $f_1(1,0,y)=0,$
4. $f_1(z+2,1,y)=y,$
5. $f_1(z+1,x+1,y)=f_1(z,f_1(z+1,x,y),y).$

(Rogers, 1967, p. 8). The first two lines here define addition, and the third line indicates that any number times zero is zero, which is the initial condition for multiplication. The fourth line indicates that for operations past multiplication, the answer obtained, when the second operand is one, is the same as the first operand; in other words, raising something to the first power does not change it. The fifth line is the workhorse of the function; it is what allows the function to be defined, given all of the initial conditions, for all natural numbers. Let us illustrate this with an example. Say we want the value of $f_1(2,2,3)$, which equals $3^2$. Here is the Ackermann evaluation of $f_1(1,3,2)$, using both Ackermann and standard notation, along with the line number in the Ackermann definition being used:

| *Ackermann* | *Std.* | *Line No.* |
|---|---|---|
| $f_1(2,2,3)$ | $3^2$ | *Given* |
| $f_1(1,f_1(2,1,3),3)$ | $3*3^1$ | 5 |
| $f_1(1,3,3)$ | $3*3$ | 4 |
| $f_1(0,f_1(1,2,3),3)$ | $3+(3*2)$ | 5 |
| $f_1(0,f_1(0,f_1(1,1,3),3),3)$ | $3+3+(3*1)$ | 5 |
| $f_1(0,f_1(0,f_1(0,f_1(1,0,3),3),3),3)$ | $3+3+3+(3*0)$ | 5 |
| $f_1(0,f_1(0,f_1(0,0,3),3),3)$ | $3+3+3+0$ | 3 |
| $f_1(0,f_1(0,3,3),3)$ | $3+3+3$ | 1 |
| $f_1(0,3+1+1+1,3)$ | $3+6$ | *Rep.* 2 |
| $(3+1+1+1)+1+1+1$ | 9 | *Rep.* 2 |

This is, to say the least, cumbersome. It can easily be seen, though, that the Ackermann function will accomodate any natural number for any of the three operands (given enough time). Its awkwardness is also what gives it its versatility, and also its power. The Ackermann generalized exponential, as well as the "bottom-up" generalized exponential to defined later, are described as "doubly nested" recursive functions. This is because, as in the derivation above, two operands need to be reduced to zero for the function to work, instead of one operand for "singly nested" functions. Because the generalized exponentials are doubly nested, a function $f(x,x,x)$ (the lack of a subscript implies that either the Ackermann function or the function to soon be defined will work) is "by-end-pieces" greater than any function using single nesting (Rucker, 1982, p.106), such as the exponential function, $f(x)=e^x$, or, the tetration or pentation functions, to be defined later. For two functions, f and g, $f>_{bep}g$ (read f is greater by-end-pieces than g) if and only if the graph of f eventually manages to get above, and stay above, the graph of g. More formally, $f>_{bep}g$ if and only if there exists some real number N such that for all $x>N$, $f(x)>g(x)$. So, in accordance with intuition, $f(x,x,x)$ is indeed a monstrously powerful function.

Under the Ackermann generalized exponential, operations after exponentiation associate from the "top down" because new copies of the first operand are always placed before the previous expression when recursion is performed. This is due to the quirk of superscription (upper-right corner) in writing exponents, which is important notationally for the definition of tetration, defined below. So, when referring to operations defined under the Ackermann generalized exponential, we will sometimes use the phrase "top down."

The other recursive function incorporating the three basic arithmetic operations, $f_2$, associates from the bottom up; it places each copy of the first operand at the end of the expression. There is no difference between the two generalized exponentials in the first three operations, addition, multiplication, and exponentiation, since addition and multiplication are commutative. Beyond exponentiation, however, there is quite a difference. Here is a recursive definition of $f_2$:

$$
\begin{aligned}
&f_2(0,0,y)=y,\\
&f_2(0,x+1,y)=f_2(0,x+1,y)+1,\\
&f_2(1,0,y)=0,\\
&f_2(z+2,1,y)=y,\\
&f_2(z+1,x+1,y)=f_2(z,y,f_2(z+1,x,y)).
\end{aligned}
$$

The only difference between the recursive definitions of the first and second generalized exponentials is in the last line. As with $f_1$, the first two lines define addition, the third sets the initial condition for multiplication, and the fourth line is the initial condition for operations higher than multiplication. The fifth line serves the same purpose as in $f_1$; the only difference is that here, new numbers in an expression are added at the end, rather than at the beginning. For example, in $f_1$, $4^4 = 4*(4^3) = 4*(4*4*4)$, while in $f_2$, $4^4 = (4^3)*4 = (4*4*4)*4$. Of course, the two results are identical, but that is only because multiplication is associative.

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"Bottom-Up" Tetration

Using the "bottom up" generalized exponential, it is possible to define operations past exponentiation. The operation immediately following exponentiation, called tetration, is defined by repeated exponentiation. "Bottom up" tetration will be notated as follows: $t_2(x,y)$, read "x tetrated to the y," or, more explicitly, "x bottom-up tetrated to the y." (Note: a subscript of two indicates "bottom up," while a subscript of one would indicate "top down.")

"Bottom-up" tetration is defined as follows. Its initial condition is that for all x, $t_2(x,1)=x$. To find $t_2(x,y)$ for any integer y>1, we can use the following recursive formula:

$$t_2(x,y+1)=t_2(x,y)^x. \tag{1}$$

To define it on integers less than 1, we must manipulate the recursive formula, which gives $t_2(x,y-1) = t_2(x,y)^{1/x}$. Table 1 shows a table of a given base, x, raised to different "tetraponents" (second operands).

$$\begin{aligned}
&\vdots\\
&t_2(x,-1)=(x^{\frac{1}{x}})^{\frac{1}{x}}\\
&t_2(x,0)=x^{\frac{1}{x}}\\
&t_2(x,1)=x,\\
&t_2(x,2)=x^x,\\
&t_2(x,3)=(x^x)^x,\\
&t_2(x,4)=((x^x)^x)^x,\\
&\vdots
\end{aligned} \qquad \text{(Table 1)}$$

With table 1, we can obtain $t_2(x,y)$ for any real x (since exponentiation is defined for all real numbers), and any integral y. However, it would be nice to define bottom-up tetration for non-integral y, as well.

Let us first determine criteria for a function serving as an extension for non-integral tetraponents, that is, for a function f, defined on all real x, such that $f(x) = t_2(b,x)$ for a constant b:

1. The function must satisfy the defining recursive equation (equation 1 for $t_2$) for all real numbers.

2. The function must be infinitely differentiable.

Bottom-up tetration happens to be relatively easy to define for non-integral tetraponents. By repeated use of the law $(a^b)^c = a^{bc}$, we can obtain the explicit formula

$$t_2(x,y)=x^{(x^{y-1})}, \tag{2}$$

for all integral y. Two examples:

$$t_2(x,3)=x^{(x^{3-1})}=x^{(x^2)}=(x^x)^x,$$
$$t_2(x,-3)=x^{(x^{-3-1})}=x^{(x^{-4})}=((((x^{\frac{1}{x}})^{\frac{1}{x}})^{\frac{1}{x}})^{\frac{1}{x}}.$$

These both agree with table 1. But does our definition meet the two criteria? First, it must satisfy the equation $t_2(x,y+1)=(t_2(x,y))^x$ for all x:

$$(t_2(x,y))^x \stackrel{?}{=} (x^{(x^{y-1})})^x=x^{(x^{y-1}\cdot x)}=x^{(x^y)}=t_2(x,y+1).$$

This works, and the function is infinitely differentiable (since the exponential function for any base is infinitely differentiable), so it seems reasonable to accept the definition in equation 2 as the extension of f(x) on non-integers.

**"Top-Down Tetration"**

The Ackermann generalized exponential has its own version of tetration. We will use the notation $t_1(x,y)$, which is read "x tetrated to the y," or "x top-down tetrated to the y." Results for tetraponents greater than one are found by using a recursive formula:

$$t_1(x,y+1)=x^{t_1(x,y)} \tag{3}$$

Results for tetraponents less than one are found by using the inverse formula,

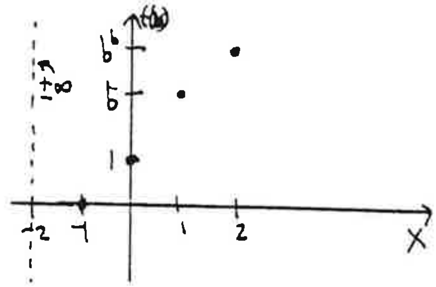
$$t_1(x,y-1)=\log_x(t_1(x,y)) \tag{4}$$

Here is a table of a given base, x, raised to several tetraponents (note that tetration is not defined for tetraponents less than or equal to negative 2):

$$\begin{aligned}
&t_1(x,-2)=\infty \ \mathit{for}\ x<1,\\
&t_1(x,-2)=-\infty \ \mathit{for}\ x>1,\\
&t_1(x,-1)=0,\\
&t_1(x,0)=1,\\
&t_1(x,1)=x,\\
&t_1(x,2)=x^x,\\
&t_1(x,3)=x^{(x^x)},\\
&\vdots
\end{aligned}$$

We must also consider the odd case of the base being equal to 1. We must define $t_1(1,1)$ to be 1. Suppose we want the value of $t_1(1,2)$. By equation 3, this equals $1^1=1$, which seems perfectly acceptable. Now suppose that we have a bad short-term memory and again want the value of $t_1(1,1)$. Even without looking at equation 4, which is what we would use (successfully) for any other base, it is apparent that any real value of $t_1(1,1)$ would work, since any number raised to the first power is one! The obvious solution is to say that $t_1(b,x)$ is undefined for b=1. Top-down tetration is also undefined for b=0, also since equation 4 will not work.

Just as in bottom-up tetration, it would be nice to define top-down tetration for non-integral tetraponents. A graph of the values of $f(x) = t_1(b,x)$, where b is a real constant (except 1) and x is an integer, is included here:




We would want our function extending top-down tetration to non-integral tetraponents to satisfy the same two criteria used for $t_2$; that is, we would want the function to (1) satisfy the defining recursive equation (equation 3), and (2) be infinitely differentiable.

$t_1$ is much harder to define for non-integral tetraponents than $t_2$, mostly because the convenient rule of multiplying exponents used in the $t_2$ definition does not apply. To illustrate the difficulties, let us try to evaluate $t_1(2,2.5)$, which, assuming the tetration function is well-behaved, we know lies somewhere between $t_1(2,2) = 4$ and $t_1(2,3) = 2^4 = 16$. If we assume an additive law analogous to the one in exponentiation ($2^{2.5} = 2^2 * 2^{.5} = 4*\sqrt{2}$), we might try to define $t_1(2,2.5)$ as $t_1(2,2)^{t_1(2,.5)}$. There are two problems with this approach: first, we don't know what $t_1(2,.5)$ is, and, perhaps more seriously, there is no equivalent additive law. To see this, consider a counterexample: $t_1(2,2)^{t_1(2,2)} = 4^4 = 64 \neq t_1(2,4) = 2^{16} = 65536$. Assuming some exotic sort of additive law, we could come up with a process which, done twice on 4, yields 16. Doubling the result seems an obvious choice. This would yield a result of $t_1(2,2.5) = 8$. However, defining tetration in this way leads to a violation of our first criterion. Following the same line of thought, $t_1(2,1.5)$ would equal $2\sqrt{2}$ (since multiplying $t_1(2,1) = 2$ by $\sqrt{2}$ twice yields $t_1(2,2) = 4$), but $2^{2\sqrt{2}} \neq 8$, the result for $t_1(2,2.5)$. It is also possible to find a number n such that $n^{n^4} = 16$, and then define $n^4$ as $t_1(2,2.5)$, but, again, the result is inconsistent with the result for $t_1(2,1.5)$.

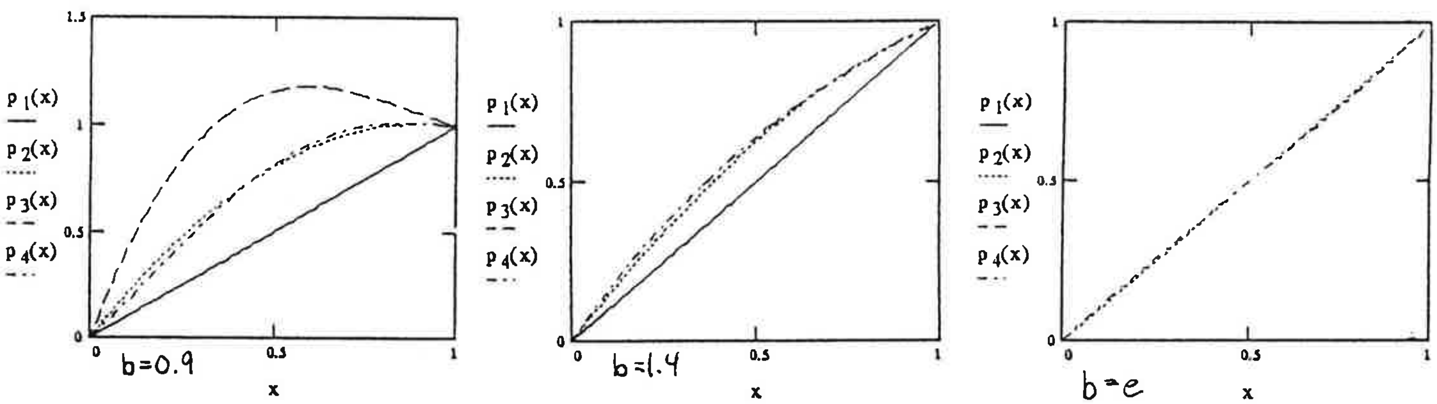
It is possible to find a sequence of polynomials which approximate a function t, where $t(x) = t_1(b,x)$, for some base b, on one unit-length interval, say, [-1,0]. This unit interval may be "copied" using equation 3 to define t(x) for all unit intervals, and, therefore, for all real numbers. For example, given some value such as t(-.5), it is possible to use the formulas $t(y+1)=b^{t(y)}$ and $t(y-1)=\log_b t(y)$ to obtain t(-1.5), t(.5), t(1.5), t(2.5), and so on.

The easiest interval on which to define the function will be [-1,0], since t(-1) = 0 and t(0) = 1 (see graph on previous page). We will define f(x) to be t(x) shifted over one unit to the right, making it a function on [0,1] (this is to make future algebra simpler).

If we assume that f is continuous, then a linear, first-degree polynomial can be defined which satisfies the equations $p_1(0) = 0$, $p_1(1) = 1$ ($p_1$ stands for first-degree polynomial approximation). Obviously, $p_1(x) = x$, but we can also go about it more systematically. Let $p_1(x) = a_1x + a_0$. Then $p_1(0) = a_0 = 0$. Also, $p_1(1) = a_1 + a_0 = 1$. Since $a_0 = 0$, $a_1 = 1$. Therefore, $p_1(x) = x$. So, to get a linear approximation, we must assume continuity and solve a simultaneous system of two equations in two unknowns.

If we further assume that f is differentiable, then we can obtain a quadratic approximation. We define a generic quadratic equation $p_2(x) = a_2x^2 + a_1x + a_0$. As before, $p_2(0) = 0$, and $p_2(1) = 1$. The third of the three necessary equations (there are three unknowns) comes from differentiating both sides of the defining equation, $f(x+1) = b^{f(x)}$. Differentiating, we get $f'(x+1) = b^{f(x)}(\log b)f'(x)$. Substituting 0 for x, we get $f'(1) = (\log b)f'(0)$. (Note: "log" means natural logarithm unless otherwise indicated.) We must also differentiate our generic quadratic equation: $p_2'(x) = 2a_2x + a_1$. Now, since $p_2(0) = 0$, $a_0 = 0$. Also, $p_2(1) = a_2 + a_1 + a_0 = 1$, so, since $a_0 = 0$, $a_2 + a_1 = 1$. Since $f'(1) = (\log b)f'(0)$, $p_2'(1) = (\log b)p_2'(0)$. We may now substitute 0 for x in our differentiated generic quadratic equation, giving $2a_2 + a_1 = (\log b)(a_1)$, and so we have three equations in three unknowns, which can be solved. The end result is $p_2(x) = (1-2/(1+\log b))x^2 + (2/(1+\log b))x$.

If we assume that f is twice-differentiable, we can obtain a cubic approximation; if we assume it is thrice-differentiable, we can obtain a quartic. In general, to obtain an nth-degree polynomial approximation, we must assume (n-1)-times differentiability, and then solve n+1 equations in n+1 unknowns. Here are graphs of $t_1(b,x)$ for three different bases: 0.9, 1.4, and e, showing the linear through fourth-degree approximations.




Suppose we do obtain an infinitely differentiable function on [0,1] which is the limit of our polynomial sequence, and we "copy" this interval on all other unit intervals (using the equation $t_1(b,x+1) = b^{t(b,x)}$), obtaining $t_1(b,x)$. Is this function necessarily infinitely differentiable everywhere? The only questionable points are the points at which x is an integer, since polynomials are always smooth, and so is $b^{p(x)}$, where p is a polynomial function, and so on. We know that $t_1$ is infinitely differentiable on [-1,0]. Is it at x = 1? On [0,1], $t_1(b,x)$ is really $b^{f(x)}$, where f(x) is the shifted function on [0,1], and, on [1,2], $t_1(b,x)$ is $b^{b^{f(x)}}$. We know that the transition point between f(x) and $b^{f(x)}$, which is the point x = 0, is infinitely differentiable, and the only difference for x=1 is that there is a "b" in front; that is, instead of f(x) we have $b^{f(x)}$, and instead of $b^{f(x)}$, we have $b^{b\hat{}f(x)}$. Raising b to the power of both sides of the "equation" indicating infinite differentiability is allowed. Likewise, testing the point x = 2, all we have to do is raise b to the power of both sides of the "equation" which indicates infinite differentiability at x=1. So all questionable points, those for which x is an integer, can be accounted for in this way.

Cumbersome as this polynomial process is, with terrible eventual higher derivatives of $f(x+1)=b^{f(x)}$, it does seem to yield an approximating sequence of polynomials. I have not been able to prove that the polynomial sequence converges, but I am almost positive that it does. First of all, the graphs look as though they converge (see the graphs on the previous page). Also, I have tried this process to define the cosine function (using the recursive equation $f(x+\pi)=-f(x)$, initial condition $f(0)=1$), and the gamma function ($f(x+1)=xf(x)$, initial condition $f(1)=1$), and they seem to converge, not only when considering the distance between successive approximations, but also they seem to converge to the cosine and the gamma functions, respectively. Moreover, I have been able to definitely prove that this process yields a convergent sequence of polynomials when applied to the function $f(x+1) = f(x)$, initial condition $f(0) = 0$. The most obvious such function is $f(x) = 0$, and it can be proven that, assuming any finite degree of differentiability, the resulting polynomial approximation is zero. Therefore, the polynomial sequence trivially converges to 0. Based on this evidence, it seems a warranted conjecture that the polynomial sequence approximating the function $f(x+1) = b^{f(x)}$, $f(0) = 0$, converges.

Even if the polynomial sequence does converge, does it necessarily provide a unique, infinitely differentiable function satisfying the initial condition and recursive function? I am not sure. Again, consider the recursive function $f(x+1)=f(x)$, initial condition $f(0)=0$. Even though its polynomial sequence converges to $f(x)=0$, there is another infinitely differentiable function where $f(x+1)=f(x)$, namely, $f(x)=\sin(2\pi x)$. Therefore, in this example, there is not a unique, infinitely differentiable function satisfying the initial condition and recursive formula. Another example worth considering is the gamma function, for which, as indicated above, this polynomial approximation procedure does seem to work. Is the gamma function the only infinitely differentiable function for which f(x+1)=xf(x)? It is unique in at least one way: it is the only such function that is log convex, meaning that the second derivative of the logarithm of the function is always positive (Artin, 1964, 13). This gives hope to our claim that there is a unique, infinitely differentiable function such that $f(x+1)=b^{f(x)}$ (a log convex function has to be very smooth), but still, we are not sure. Perhaps, though, the function for tetration yielded by the polynomial sequence is, in some way, the "simplest" such function. This was the case in the example of f(x+1)=f(x), yielding f(x)=0, instead of some odd sinusoidal function. In any case, we do have a definition for $t_1$ which meets our two criteria.

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**"Bottom-Up" Pentation**

It is also possible to define pentation, the operation after tetration, under the second version of the generalized exponential. We should probably call this function $p_2(x,y)$, but, as we will not discuss $p_1$ (pentation under the Ackermann exponential), and as $p_2$ might be confused with a polynomial approximation of degree two, we will simply call bottom-up pentation $p(x,y)$ (read x pentated to the y).

Now, by the second generalized exponential function, $p(x,y+1) = t_2(x,p(x,y))$, and $p(x,1)=x$. Since $t_2(x,y) = x^{x^{(y-1)}}$,

$$p(x,y+1)=p(x,y)^{(p(x,y)^{x-1})}. \tag{5}$$

How are we to define this function for non-integral "pentaponents?" We could use the same approach as we did with $t_1$, differentiating equation 5 until there are enough differential equations to define successive degree polynomials. However, the derivatives of that function would be extremely hard to manage. We shall try to simplify the problem a bit.

Let us first develop a table for pentation. We know that $p(x,1)=x$ (the initial condition) and $p(x,2)=x^{x^{(x-1)}}$. To get $p(x,3)$:

$$p(x,3)=(x^{(x^{x-1})})^{((x^{(x^{x-1})})^{x-1})},$$

which simplifies to

$$p(x,3)=x^{(x-1)(1+x^{x-1})}.$$

Using equation 5 repeatedly, then simplifying, we can get the following table:

$$\begin{aligned}
&p(x,1)=x,\\
&p(x,2)=x^{(x^{x-1})},\\
&p(x,3)=x^{(x^{(x-1)(1+x^{x-1})})},\\
&p(x,4)=x^{(x^{(x-1)(1+x^{x-1}+x^{(x-1)(1+x^{x-1})})})},\\
&\vdots
\end{aligned}$$

This table suggests the following formula:

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$$p(x,y)=x^{(x^{(x-1)a_y})},$$

where a is a sequence such that

$$a_1=0,\ a_{n+1}=a_n+x^{(x-1)a_n}. \tag{6}$$

This can be proven on the natural numbers by mathematical induction. The first case, n being 1, fits with the definition of p, since p(x,1) = x, and $x^{x\hat{}\{(x-1)(0)\}}=0$. Assuming the formula is true for n,

$$\begin{aligned} p(x,n+1)=t_2(x,x^{x^{(x-1)a_n}}) &=(x^{x^{(x-1)a_n}})((x^{x^{(x-1)*a_n}})^{x-1}) \\ &=x^{x^{(x-1)a_n}\cdot x^{(x-1)x^{(x-1)a_n}}} \\ &=x^{x^{(x-1)a_n+(x-1)x^{(x-1)a_n}}} \\ &=x^{x^{(x-1)(a_n+x^{(x-1)a_n})}}. \end{aligned}$$

By induction, this shows that the sequence in equation 6 is correct.

A natural step would be to define $a_n$ on non-integers. To do this, we may use the same procedure as when defining $t_1$ on non-integers. This will be easier than trying to define the whole function, defined by the recursive formula in equation 5. This is primarily because, as when defining $t_1$, we can define a function f on an interval [0,1] such that f(0)=0 and f(1)=1, which greatly simplifies the algebra.

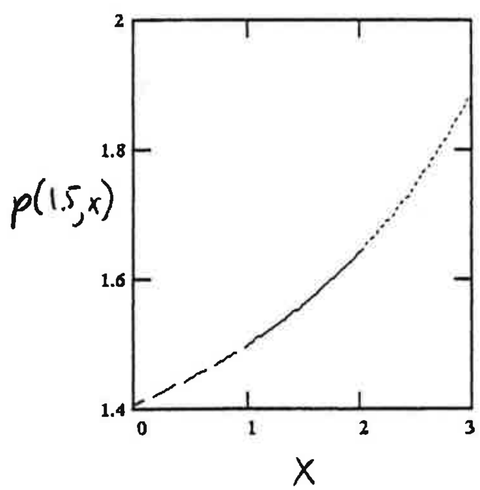
To obtain a quadratic approximation to f, we again assume differentiability and take the derivative of the recursive formula in equation 6, replacing $a_n$ with f(x) and x with b (so we don't have two different x's). Since $f(x+1)=f(x)+b^{(b-1)f(x)}$, $f'(x+1)=f'(x)+b^{(b-1)f(x)}(\log b)f'(x)$, which, when we plug in zero for x, and 0 for f(x), yields f'(1)=f'(0)(1+log b). Solving the system of three simultaneous equations, we get that $p_2(x)=(1-2/(2+\log b))x^2+(2/(2+\log b))x$, where $p_2$ is the quadratic approximation to f. Of course, just as in $t_1$, we can take higher and higher derivatives of f to obtain closer polynomial approximations.

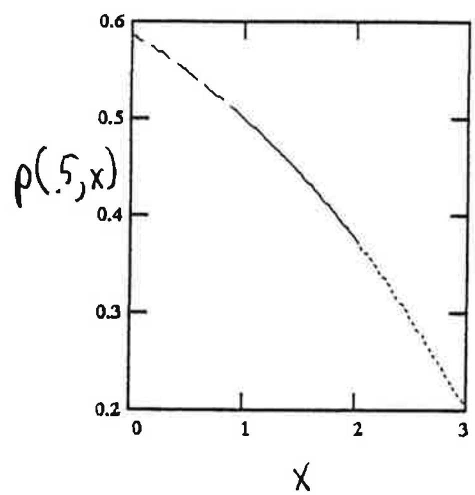
Once we have f, the limit of the approximating polynomial sequence, on [0,1], we can define f on all unit intervals after [0,1] using the formula $f(x+1)=f(x)+b^{(b-1)f(x)}$. Somewhat of a drawback is that there is no simple algebraic inverse formula which defines f(x-1) in terms of f(x). However, it can be found approximately using numerical methods.

Once we have defined f for all real numbers, we may define bottom-up pentation as follows:

$$p(x,y)=x^{x^{(x-1)f(y-1)}}.$$

Here are graphs of p(b,x) for two bases, b=.5 and b=1.5, where x ranges from 0 to 3.



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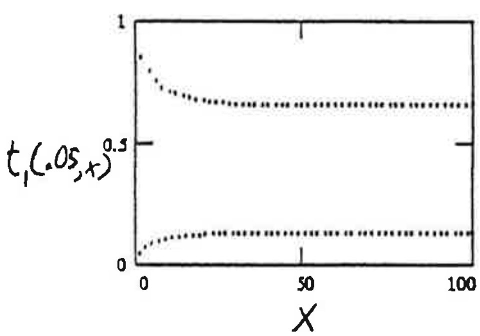
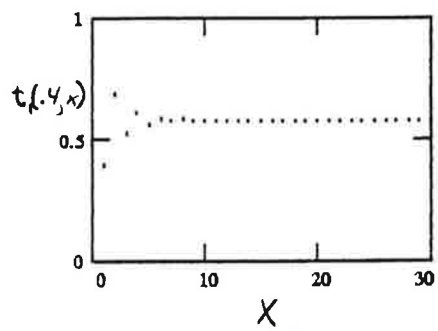
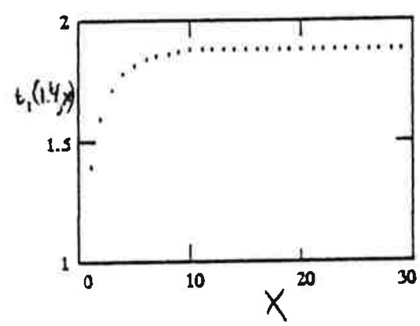
## Interesting Properties of "Top-Down" Tetration

If we consider the exponential function on the real numbers, $f(x)=b^x$, for some positive, real base b, and analyze what happens for higher and higher exponents, we find that if $b>1$, $f(x)$ increases without bound; if $b=1$, the function $f(x)=1$ for all x; and if $b<1$, $f(x)$ goes to zero.

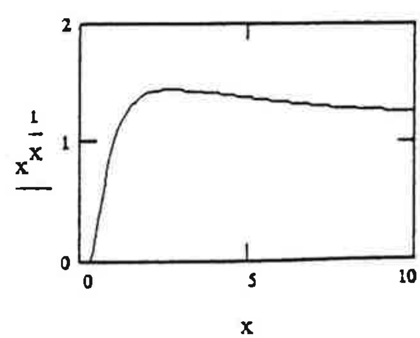
One would expect the same thing to occur for top-down tetration, but it does not. Of course, for large values of b, the function $t_1(b,x)$ increases without bound. However, tetrating numbers to high tetraponents on a computer yields surprising results: if $b\in(1,\sim1.444]$ ($\sim$ meaning approximately), $t_1(b,x)$ approaches a limit from below; if $b\in[\sim.06,1)$, $t_1(b,x)$ oscillates for a time, but finally settles on a limit; and most interestingly, if $b\in(0,\sim.06)$, $t_1(b,x)$ oscillates more vigorously, then actually approaches two limits as x goes to infinity. Here are the graphs of $t_1(0.05,x)$, $t_1(0.4,x)$, and $t_1(1.4,x)$ to illustrate:




To obtain the greatest base b for which $\lim_{x\to\infty}(t_1(b,x))$ converges, we may consider the solutions to the equation $b^x=x$, since successive answers must equal each other. Solving for b, we get $b=\sqrt[x]{x}$. Here is a graph of that function:



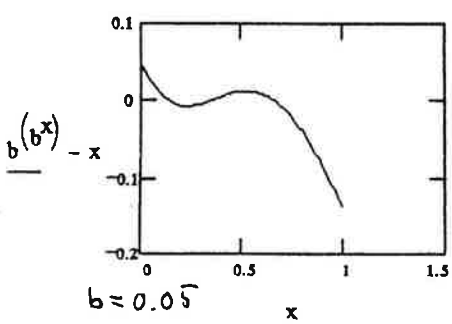
The greatest b for which $\lim_{x\to\infty}(t_1(b,x))$ converges to one limit is the maximum value of b on the above graph. To find the maximum value of this function, we can take the derivative of $b^x=x$ with respect to x, obtaining $b^x(\log b)=1$. Substituting x for $b^x$, we get $x(\log x^{1/x})=1$. By the multiplicative rule for logarithms, log x=1, so x=e. Therefore, $b=\sqrt[e]{e}$.

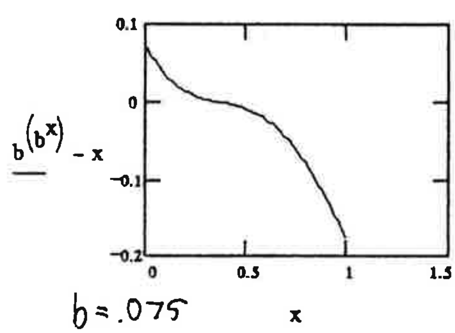
Finding the least b for which $\lim_{x\to\infty}(t_1(b,x))$ converges to one value, or the greatest b for which it converges to two values, is a bit more difficult. The desired b must satisfy the equation

$$b^{(b^x)}-x, \qquad (7)$$

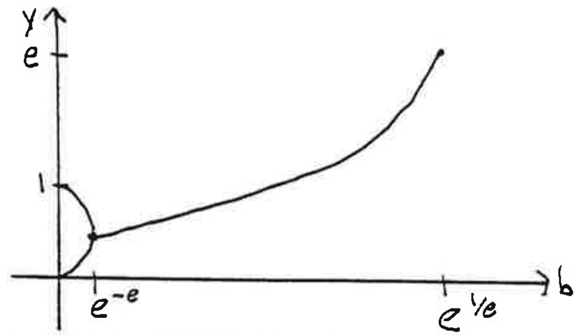
since we want $t_1(b,x)$ to equal $t_1(b,x+2)$. It happens that there are three sets of (b,x) which satisfy this equation on (0,-.06). First, there is the pair (b,x) which satisfies the equation $b^x=x$. There are also two pairs, the ones we want, which are the limits of the tetration function, $\lim_{x\to\infty}(t_1(b,x))$. We want the greatest b such that we get three roots, or the least b such that we get one root.

To do this, it would be helpful to look at the graph of $f(x)=b^{(b^\wedge x)}-x$ (zeroes of this function would indicate roots of equation 7) for values of b that we know are less than the desired b, and values of b that we know are greater than the desired b. Here are graphs of f(x), where b=0.05 and b=0.075:

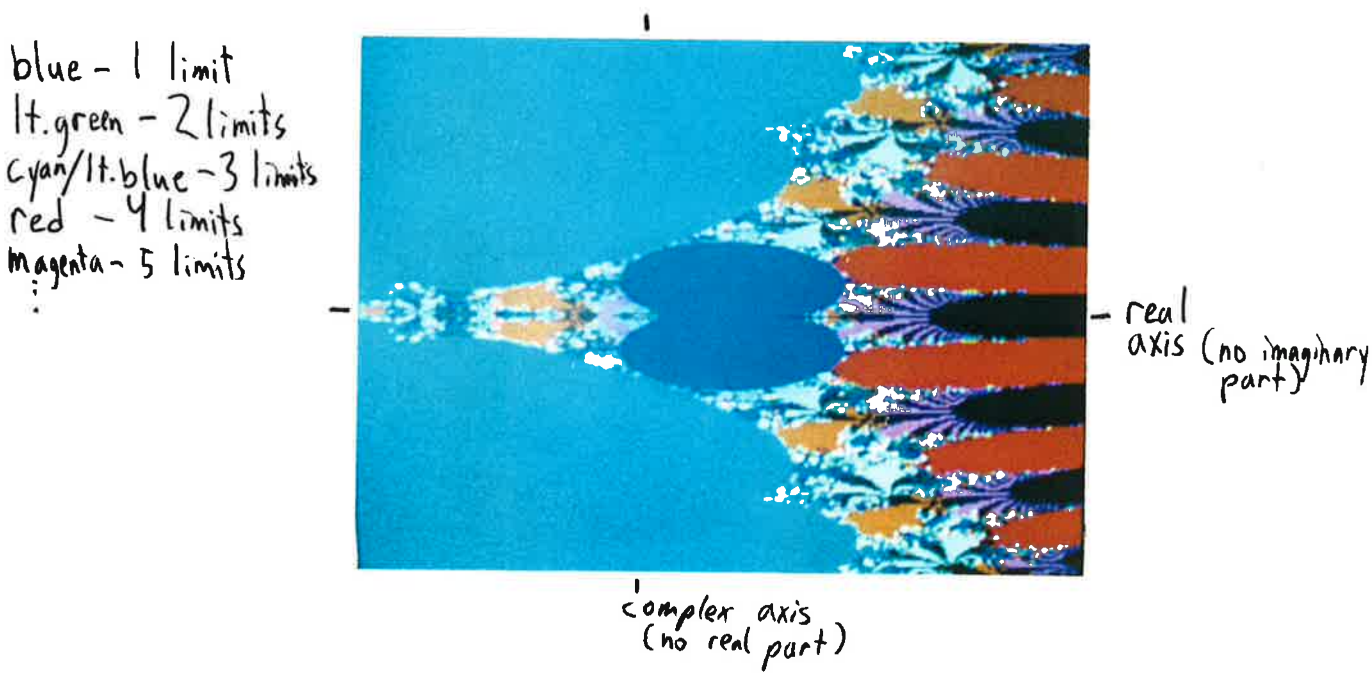



Notice that, in the second graph, the slope where f(x) crosses the x-axis is negative, while in the first graph, the slope at the middle zero of f(x) is positive. It seems fair to assume that, if b is the first b for which there is only one root, the zero would be at a horizontal inflection point (at which f(x), along with its first two derivatives, would equal zero). If f(x)=0 and f'(x)=0, then we will have two equations in two unknowns. Unfortunately, they must be solved numerically, but we still get an expression involving e: the desired b equals $e^{-e}$. Here is a graph showing $y=\lim_{x\to\infty}(t_1(b,x))$:




It is even more interesting that for some complex values of b, the tetration function has several limits (not only two!). Here is a graph on the complex plane showing the number of limits a given base has when it is taken to higher and higher tetraponents. Each color represents a different number of limits; for example, the large, blue region near the center represents one limit (the real line is the line of symmetry).



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**Conclusion**

The operations of tetration and pentation are indeed fascinating, and I have only investigated a few aspects of them. I have found an algebraic way to define bottom-up tetration, but, as for top-down tetration and bottom-up pentation, I am not sure that the respective polynomial sequences converge to unique functions. It would have been much better to find one expression which works (or something like a power series), and that is one possible avenue of further investigation. It would also be interesting to somehow define top-down pentation, as well as operations past pentation under both generalized exponentials. Also, inverse operations after exponentiation, such as inverse tetration, for example, would be interesting to investigate.

Yet another unexplored topic would be to define the generalized exponentials for non-integral first operands; that is, to define operations <u>between</u> addition and multiplication, et cetera, and, perhaps, to define operations before addition. Then, we would be further generalizing a "generalized" exponential.

Another point which belongs here is the question of why tetration, pentation, etc. are not "useful" in the real world, while the operations of addition through exponentiation are. Most mathematicians probably go through their whole lives without knowing about operations past exponentiation (not particularly that they should). It is possible that one day, we will discover these operations' physical significance (if they have any); right now, they have none. However, that does not mean that they should not be studied.

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## BIBLIOGRAPHY

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