Origami is an art form with roots in Asia more than 1,000 years in the past, and likely coinciding with the invention of paper nearly 2,000 years ago. The Japanese word ‘origami’ literally means ‘fold, paper.’ Interest in the mathematics of origami arose only in the last century, and a focus on “computational origami” only since the 1990s.

In this part, I have selected three approachable topics. The first two concern “flat foldings,” a specialized form of origami. We first concentrate on single-vertex flat folds that, although not very exciting as origami, include some beautiful mathematical regularities. These regularities will help us in the next chapter explain the amazing “Fold and One-Cut” theorem, which is perhaps the prettiest result so far obtained in mathematical origami. And we close this Part of the book with another surprising but more specialized theorem, the “Shopping Bag” theorem.



# 4 Flat Vertex Folds

Although an origami folding generally produces a 3D object, such as the ubiquitous crane, intermediate stages of the folding are often *flat*, that is, parallel layers of paper squashed into a plane, as in Figure 4.1. In fact, flat origami as an end-product is its own well-developed art form.

In this chapter, we examine some of the surprising regularities present in flat origami, and then touch on the perhaps even more surprising technical unknowns lurking in a problem as commonplace as folding a map.

## 4.1 Mountain and Valley Creases

When you fold a sheet of paper in half, you create a straight-line *crease* that extends from one edge of the paper to an opposite edge. A crease snaps fibers in the paper, which is why the crease imprint remains after the creasing pressure is released, and why you cannot erase a crease completely by uncreasing – the fibers remain broken. Origami creases need not in general extend from edge to edge of the paper being folded. With some care, you can crease a line segment in the interior of the paper, with neither endpoint at the paper edge.

Creases come in two varieties: those created by a *mountain fold* and those by a *valley fold*, with natural meanings; see Figure 4.2. Traditionally, valley folds are indicated in origami diagrams as dashed lines – – – – –, and mountain folds by a dash-dot pattern, – · – · – · –. Because these patterns are easily confused by the eye, we opt for the unconventional red for mountain and green for valley. (Memory aide: red sunset hitting peaks, lush green valleys.) Whether a crease represents a mountain or a valley fold depends on the point of view: From the underside, a mountain fold becomes a valley fold, and vice versa.

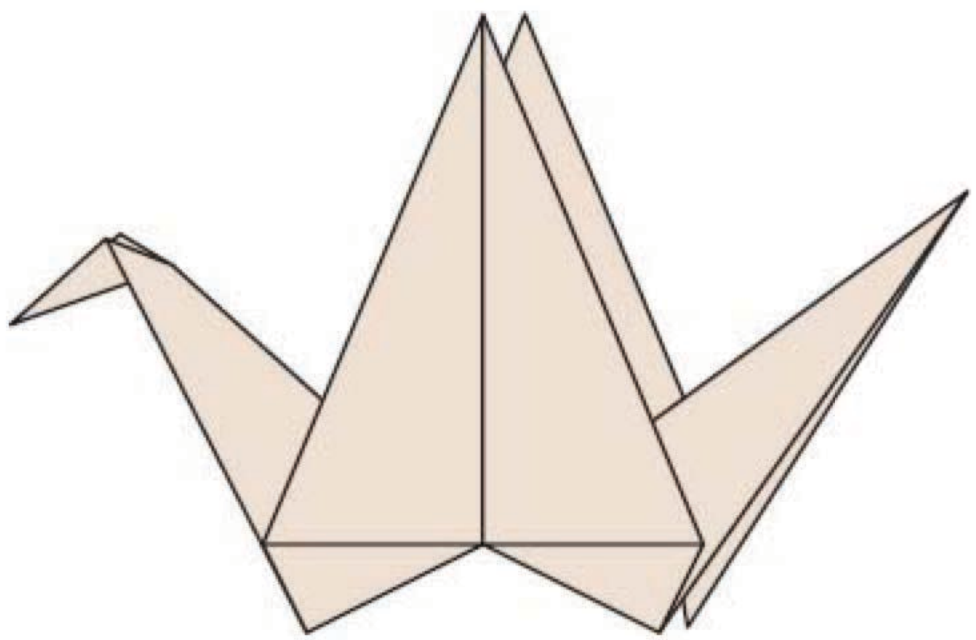


Figure 4.1. The standard origami crane, shown as a flat folding, before wings flap into 3D.

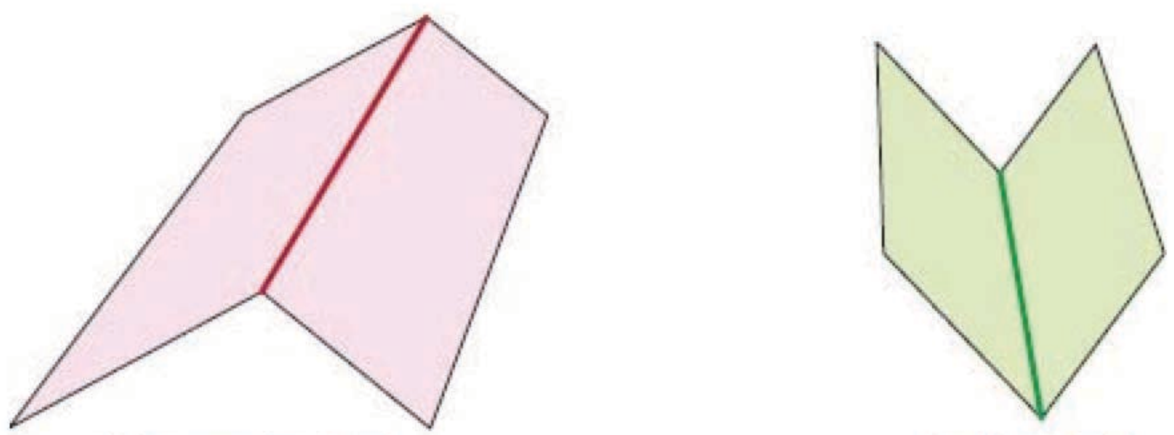



Figure 4.2. Mountain and valley folds.

## 4.2 Single-Vertex Flat Folds

There is already a rich mathematical structure in one of the simplest flat origami constructions: a flat folding containing a single vertex. A *vertex* in an origami construction is any point not on the boundary of the paper at which two or more creases meet. A simple example is the result of folding a sheet of paper in half twice: once top-to-bottom, and then left-to-right, which produces a vertex at which four creases meet; see Figure 4.3, in which the two sides of the paper are colored different shades.

### Box 4.1: Folding Creases

Folding a crease that goes straight through a vertex is as easy as folding a sheet of paper in half. Folding a crease that stops at a vertex requires a somewhat different technique. One method is to fold the crease lightly right through the vertex, and then only firm up the crease (perhaps by pressing against a table) for the desired half. The method I use myself is to first draw the crease on the mountain side with a ruler. Then I hold the paper in the air and pinch at several spots along the crease between my thumb and forefinger, up to but not through the vertex. Only once it has been precreased in this way do it set it on a table and sharpen the crease, either by sliding my thumbnail along it, or – better – pressing the edge of a ruler along the crease.

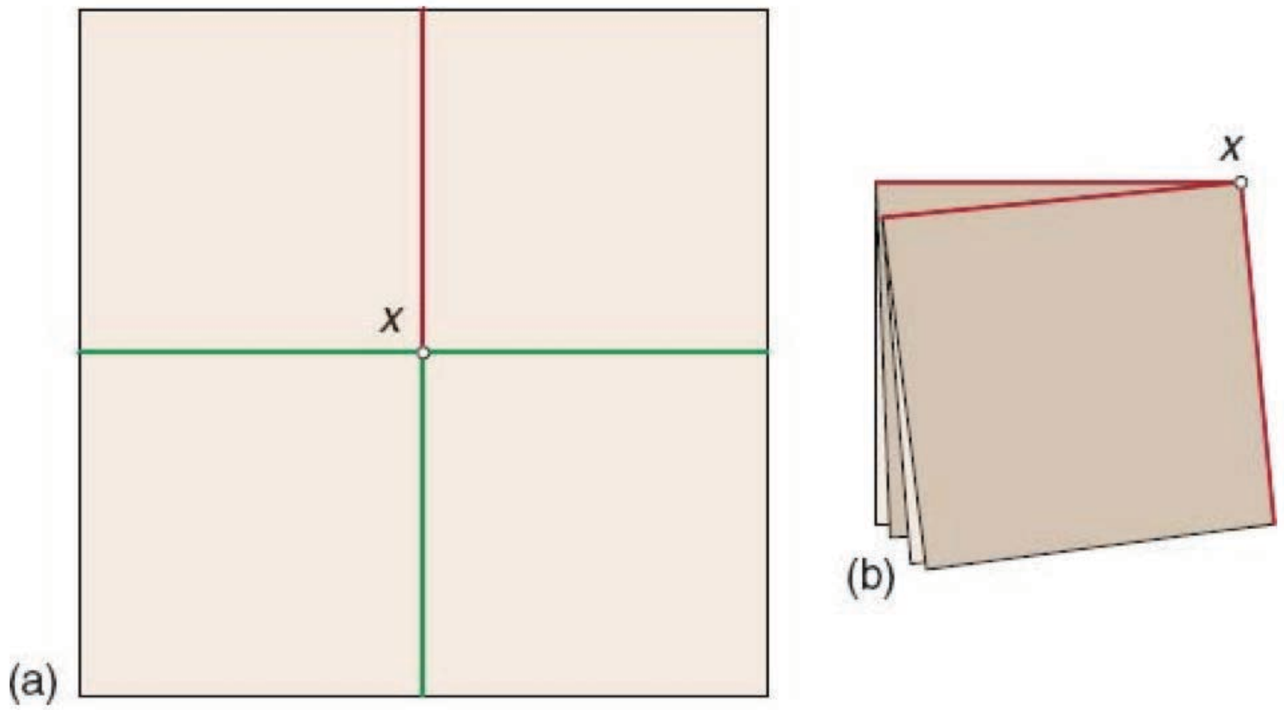



Figure 4.3. Degree-4 vertex: (a) Mountain/Valley creases on lighter side of paper; backside is darker. (b) Flat folding. The three valley creases become mountain creases on the darker side.

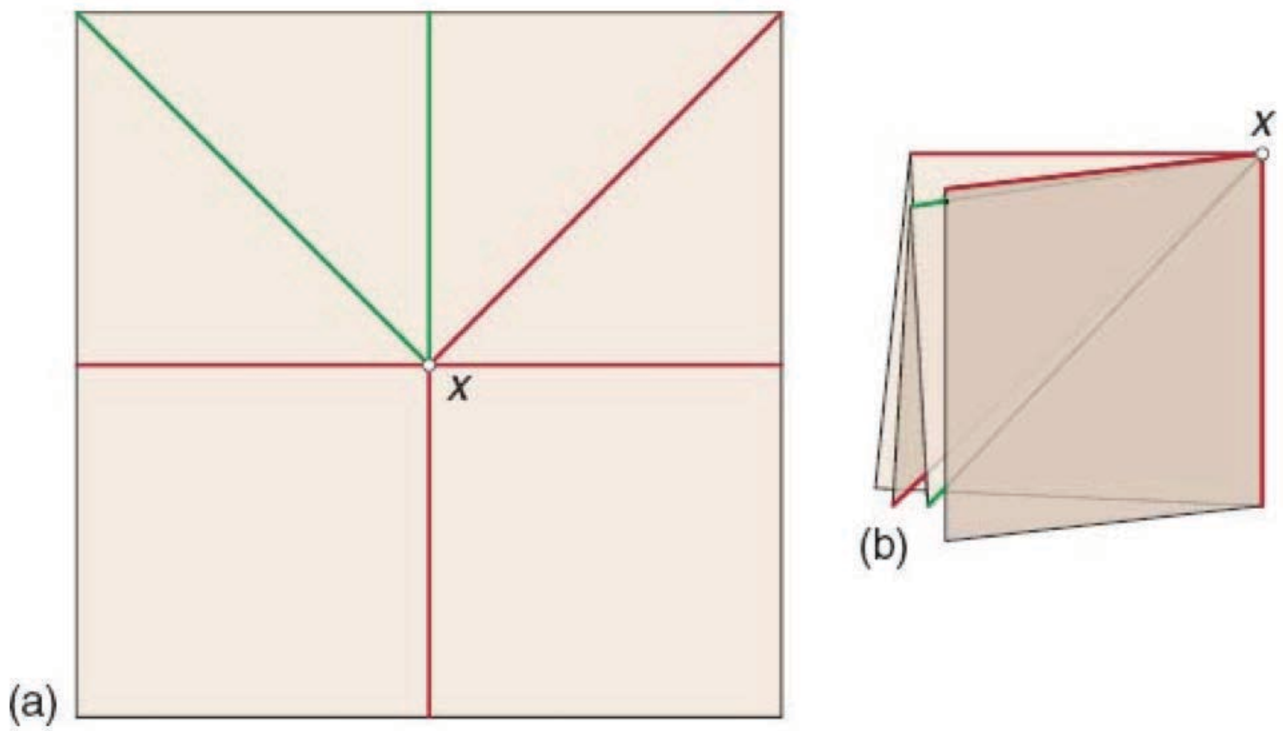



Figure 4.4. Degree-6 vertex: (a) crease pattern; (b) folding. Some sheets are shown partially transparent.

More complicated examples can be made by terminating a crease at the vertex, for example, as in Figures 4.4. and 4.5.

**Exercise 4.1 (Practice)** *Four Mountain Creases*. Create four mountain creases meeting at a central vertex, as shown in Figure 4.6, as follows. Fold a piece of paper in half, top to bottom. Now unfold completely, and fold it in half, left to right so that the two perpendicular creases are both mountain creases (or valley creases from the opposite side). Open the paper again. Convince yourself by manipulation that the paper cannot fold flat with just those four creases mountain-folded and meeting at the central vertex $x$ (as they do in Figure 4.3(b)).

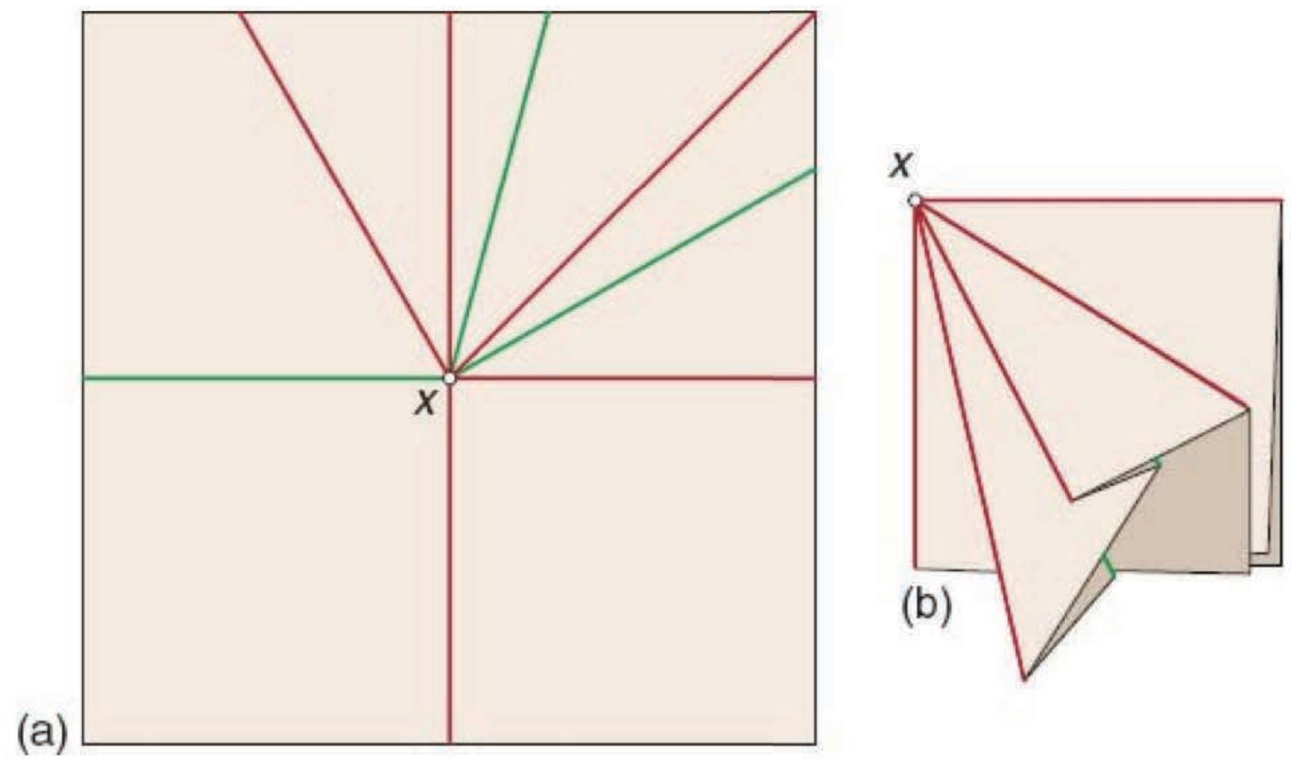



Figure 4.5. Degree-8 vertex: (a) crease pattern; (b) folding.

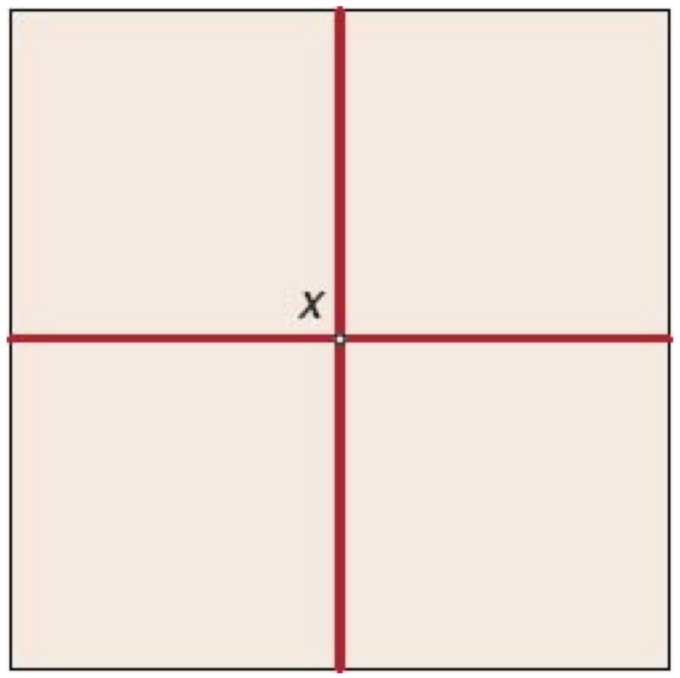



Figure 4.6. Four mountain creases meeting at vertex $x$. (Exercise 4.1)

Is there any pattern to the single-vertex flat foldings we've examined so far? I encourage the reader to experiment with sheets of paper and formulate conjectures. Perhaps the first regularity to become apparent is that the number of creases meeting at the vertex must be even in order for the pattern to fold flat: 4 in Figure 4.3, 6 in Figure 4.4, 8 in Figure 4.5. And if we consider the midpoint of the single crease formed by folding the sheet in half, a special type of vertex where two mountain folds meet along the same line (collinearly), then again there must be an even number: 2.

Indeed this regularity holds universally:

**Theorem 4.1 (Even Degree)**

*A vertex in a flat folding has even degree.*

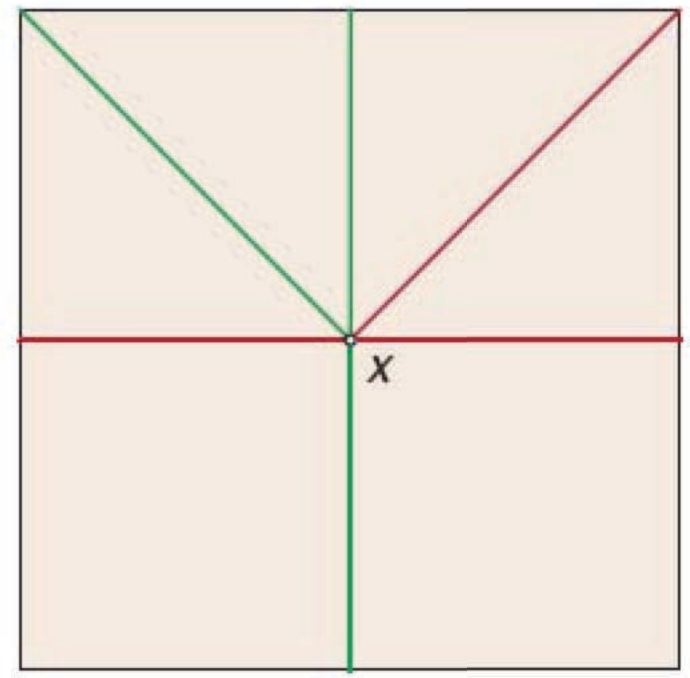



Figure 4.7. The crease pattern of Figure 4.4(a) with different mountain/valley folding labels.

In this case, *degree* has nothing to do with angular measure, but rather is the technical term for the number of creases coming into (*incident to*) the vertex. This theorem will turn out to be a consequence of a deeper regularity that we will see in the next section. Exercise 4.1 shows there must be more here, because there four creases would not fold flat. Another clue is the six-crease example in Figure 4.7, where the creases follow the same lines as in Figure 4.4 but with a different mountain/valley folding. Try as you might, you cannot fold this diagram flat. There must be both some imbalance between mountain and valley folds, and some near-balance. The regularity here is captured in a beautiful theorem named after the two people who first discovered it (independently of one another) in the 1980s, Jun Maekawa and Jacques Justin.

## 4.3 The Maekawa-Justin Theorem

## Theorem 4.2 The Maekawa-Justin Theorem

*If M mountain creases and V valley creases meet at a vertex of a flat folding, then M and V differ by 2: either $M = V + 2$ or $V = M + 2$.*

We check our examples so far (Table 4.1.), verifying that they do indeed satisfy Theorem 4.2. We now prove Theorem 4.2.

Let's start with a circular piece of paper (Figure 4.8(a)) so we are not distracted by the corners, which are irrelevant to what happens in the *neighborhood* of the single central vertex. Now we consider an arbitrary single-vertex flat folding of the paper; our goal is to prove that $M$ and $V$ differ by 2. Lay the folding flat, as in Figure 4.8(b). Now look at the side of the folded paper toward the vertex inside to see a closed zig-zag path of circular arcs, as depicted in (c) of the figure. Each arc is a piece of the circular boundary flattened between two creases, which, viewed edge-on, appears as a straight segment. Select any point of the path not directly at a crease, for example, point $p$ in (c), and imagine walking toward the right. Let's view your direction of travel as a vector (see Chapter 1, Box 1.3, on vectors). Then the start direction vector points at angle 0° in the standard coordinate system, in which angles are measured counterclockwise from the positive x-axis, which points toward the right.

**Table 4.1.** The number of mountain and valley creases ($M$ and $V$ respectively) in our examples, checking the Maekawa-Justin theorem (Theorem 4.2).

| Figure | $M$ | $V$ | $M-V$ | Theorem 4.2 satisfied? |
|---|---|---|---|---|
| Figure 4.3 | 3 | 1 | 2 | ✓ |
| Figure 4.4 | 4 | 2 | 2 | ✓ |
| Figure 4.5 | 5 | 3 | 2 | ✓ |
| Figure 4.6 | 4 | 0 | 4 | × |
| Figure 4.7 | 3 | 3 | 0 | × |

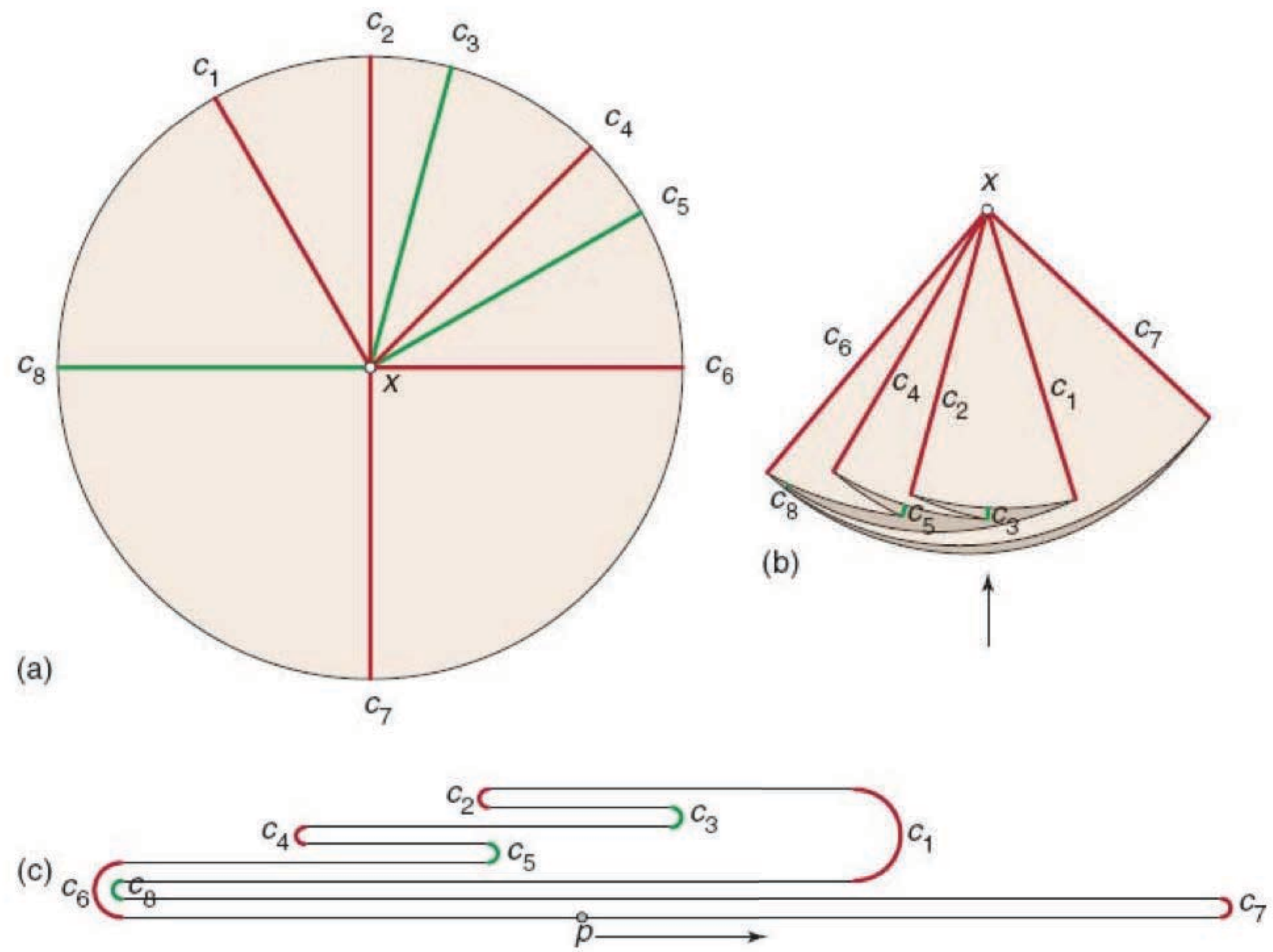


Figure 4.8. The example of Figure 4.5 revisited: (a) Crease pattern on circular paper. The eight creases are labeled $c_1, \ldots, c_8$. (b) Flat folding. (c) Expanded view looking from folded boundary toward vertex. Sharp turns at creases are shown as circular arcs to illustrate the nesting. Starting direction vector from $p$ toward right.

Each mountain fold you encounter in your walk rotates your direction vector through $+180°$ (+ meaning counterclockwise), and each valley fold rotates your vector through $-180°$ (− for clockwise). Although it is true that rotation by $+180°$ and by $-180°$ bring the vector to the same final heading – exactly opposite to the heading before rotation – the intermediate headings are different. For mountain turns, the headings point to the exterior of the folding; for valley turns, the headings point to the inside of the construction.

Now, we know that by the time we return to the starting point $p$ after traversing the entire diagram in (c), we approach $p$ from the left heading right, so again the vector has direction 0°, which is the same as 360°. In other words, we must twist a total of a full 360° by the time we return to start.

So we must have:

$$M \cdot 180° + V \cdot (-180°) = 360°$$

Dividing through by 180° leads to $M - V = 2$. Remembering that $M$ and $V$ are two sides of the same coin, we know that flipping the paper over in Figure 4.8(a) would interchange the roles of $M$ and $V$, and we'd reach the conclusion that $V - M = 2$. Combining both possibilities into one phrase: $M$ and $V$ differ by 2. That is the exactly the claim of the theorem; so we have proved Theorem 4.2.

The Maekawa-Justin theorem easily implies the Even-Degree Theorem (Theorem 4.1). Suppose $M = V + 2$. Then:

$$M + V = (V + 2) + V = 2V + 2 = 2(V + 1)$$

and so the total number of creases $M + V$ into a vertex $x$ (as in Figure 4.8a,b) is even. The same logic applies when starting with $V = M + 2$ and reaches the same conclusion: $M + V$ is even.

Most theorems have many proofs, often starting from different background assumptions. An alternate proof of Theorem 4.2 using polygons is presented in Box 4.2.

## Box 4.2: Proof of Maekawa-Justin Theorem via Polygons

The following proof was found by Jan Siwanowicz when he was a high-school student. The starting point of his proof is another theorem: The sum of the internal angles at the $n$ vertices of a polygon is $(n - 2)180°$. (This in turn follows from the theorem that every polygon can be partitioned by diagonals between its vertices into $n - 2$ triangles, so that the total internal angle is that of $n - 2$ triangles, each of which has angle sum 180°.) The idea is to view the zig-zag path in Figure 4.8(c) as a squashed polygon, as in Figure 4.9, which is closer to how it looks with sharp creases. If we imagine compressing this polygon completely flat, all the mountain vertices have an internal angle near 0°, and all the valley vertices have an internal angle near 360°. So the total internal angle sum after complete flattening is:

$$M \cdot 0° + V \cdot 360°$$

and this must equal $(n - 2)180°$, where $n$ is the total number of vertices of the polygon. In this construction, each vertex derives from a crease, so $n = M + V$. Therefore:

$$V \cdot 360^\circ = (M + V)180^\circ$$

and dividing by $180^\circ$ yields $2V = M + V$ or $M - V = 0$.

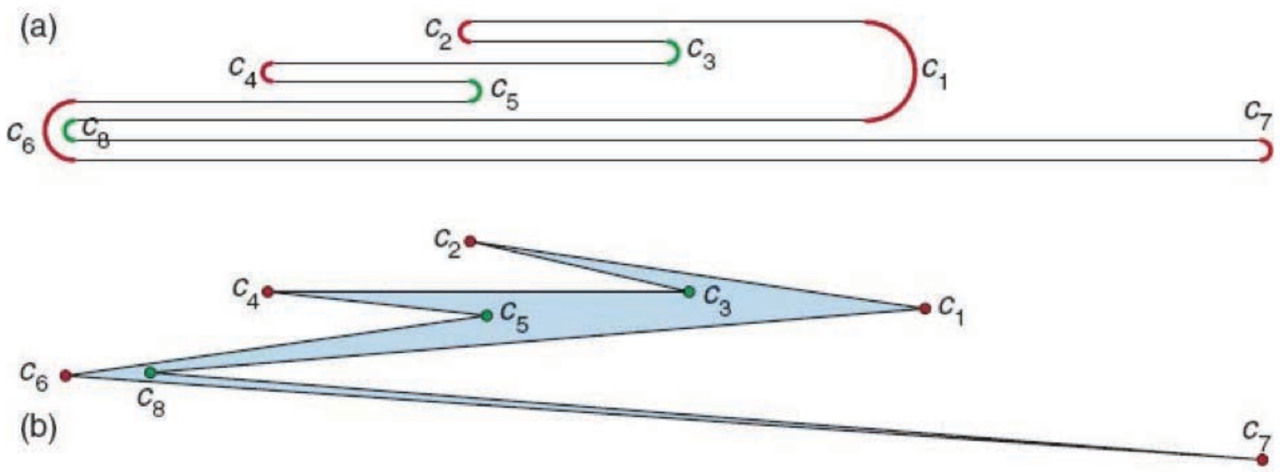



Figure 4.9. An 8-vertex polygon (b) corresponding to Figure 4.8(c), repeated as (a) here, with 5 mountain vertices $\{c_1,c_2,c_4,c_6,c_7\}$ and 3 valley vertices $\{c_3,c_5,c_8\}$.

**Exercise 4.2 (Practice)** *Maekawa-Justin Theorem.* Exercise 4.1 argued that four mountain creases lead to an unflattenable vertex. Add additional creases to Figure 4.6 so that it can flatten, and verify the Maekawa-Justin Theorem (Theorem 4.2) for your construction.

## 4.4 The Local Min Theorem

The pattern in Figure 4.10(a) shows that we still haven't plumbed the depths of single-vertex flat foldings fully. It satisfies the Maekawa-Justin Theorem (Theorem 4.2) with $M = 4$ and $V = 2$, and therefore satisfies the Even-Degree Theorem (Theorem 4.1) with degree 6. And yet, if you try to fold it flat, you will see it is impossible. Why? The essence of the impediment is that a 40° pie-slice *wedge* delimited by two valley folds is surrounded by larger angles on either side – 70°. This forces paper to pass through itself, as depicted in (b) of the figure. Whenever we have such a pattern of consecutive wedge angles: {large, small, large}, the folds delimiting the central wedge cannot both be valley, nor can both be mountain: One must be mountain and the other valley. The central angle is called a *local min*, because locally – that is, in its immediate neighborhood – it is a minimum angle, smaller than its neighbors to either side. We can phrase this condition in a theorem as follows:

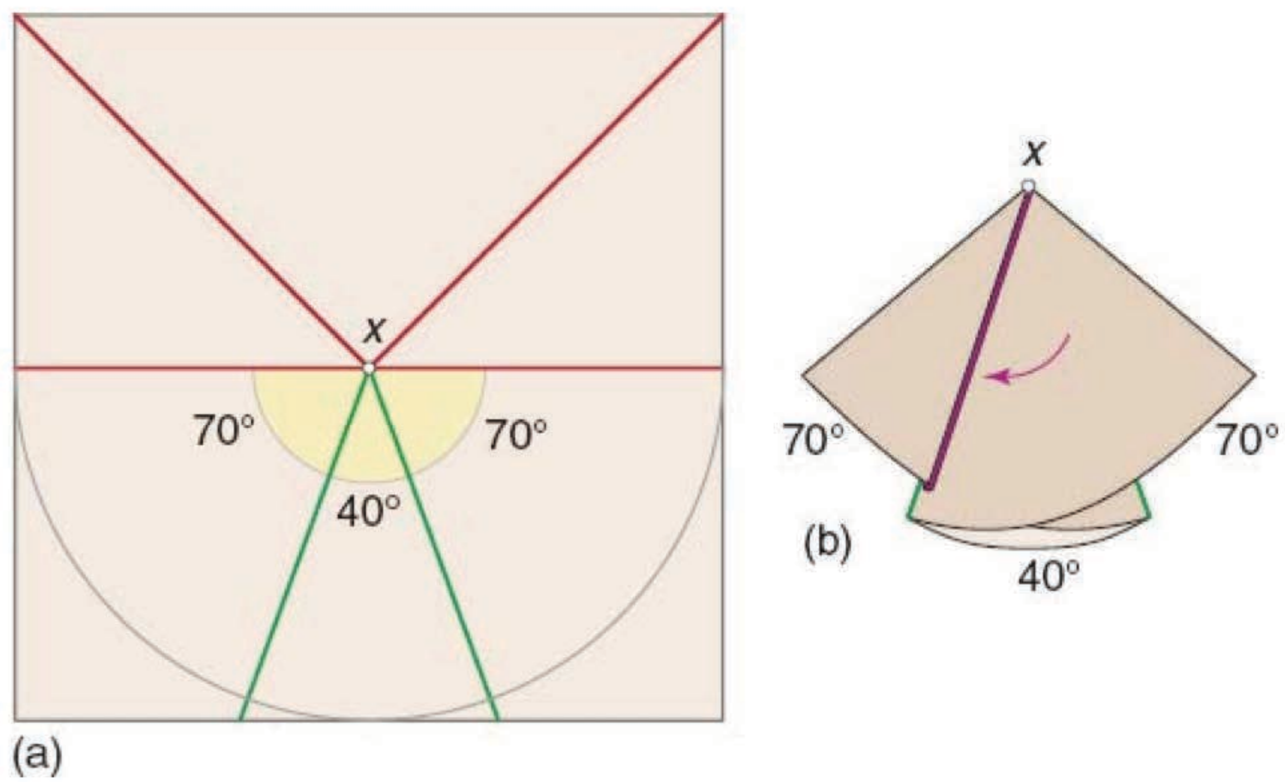



Figure 4.10. (a) A crease pattern that cannot fold flat. (b) Attempting to fold the 40° wedge results in paper penetrating itself. Here we've restricted the folding to the semicircle (a) to make the angular relationships clear.

**Theorem 4.3 (Local Min)**

*In any flat folding, any wedge whose angle is a local min must be delimited* *by one mountain and one valley fold.*

**Exercise 4.3 (Practice)** *Three Theorems Check*. Check which of Theorems 4.1 (Even Degree), 4.2 (Maekawa-Justin), and 4.3 (Local Min) are satisfied by the crease pattern in Figure 4.11.

The three regularities we've uncovered so far are what mathematicians call *necessary conditions*: Every single-vertex flat folding necessarily satisfies them. But they may or may not be *sufficient conditions*: conditions on the crease pattern which, if satisfied, imply the diagram can be folded flat. Indeed our three conditions, in pairs or even all three together, are not sufficient conditions. The holy grail in mathematics is a set of necessary and sufficient conditions, which then completely characterize the situation. For single-vertex flat folds, these are embodied in the Kawasaki-Justin Theorem.

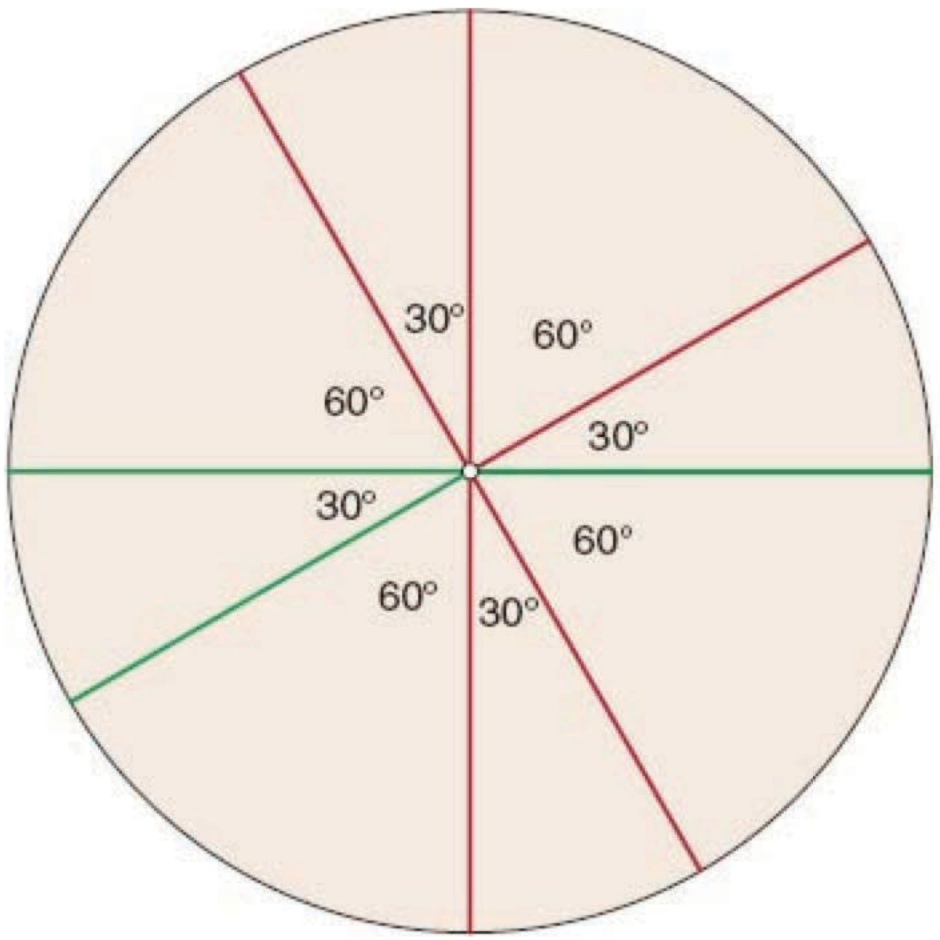



Figure 4.11. A single-vertex crease pattern for checking. (Exercise 4.3)



## 4.5 The Kawasaki-Justin Theorem

The Local-Min Theorem (Theorem 4.3) indicates that the measures of the wedge angles defined by the crease pattern are important. Let us call the wedge angles around the vertex in sequential order, $\theta_1, \theta_2, \ldots, \theta_n$. We know from the Even-Degree Theorem (Theorem 4.1) that $n$ is even, because an even number of creases determine an even number of wedges. We also know that:

$$\theta_1 + \theta_2 + \cdots \theta_n = 360°$$

because the angles completely surround the vertex. The Kawasaki-Justin Theorem claims that a simple condition on the angles, completely ignoring the mountain-valley pattern, provides necessary and sufficient conditions for flat foldability:

### Theorem 4.4 (Kawasaki-Justin)

*A set of an even number of creases meeting at a vertex folds flat if, and only if, the alternating sum of the determined wedge angles is zero:*

$$\theta_1 - \theta_2 + \theta_3 - \theta_4 + \cdots + \theta_{n-1} - \theta_n = 0°$$

The term *alternating sum* means that every other term has opposite sign: The odd terms $\theta_1, \theta_3, \theta_5, \ldots$ are added and the even terms $\theta_2, \theta_4, \theta_6, \ldots$ are subtracted.

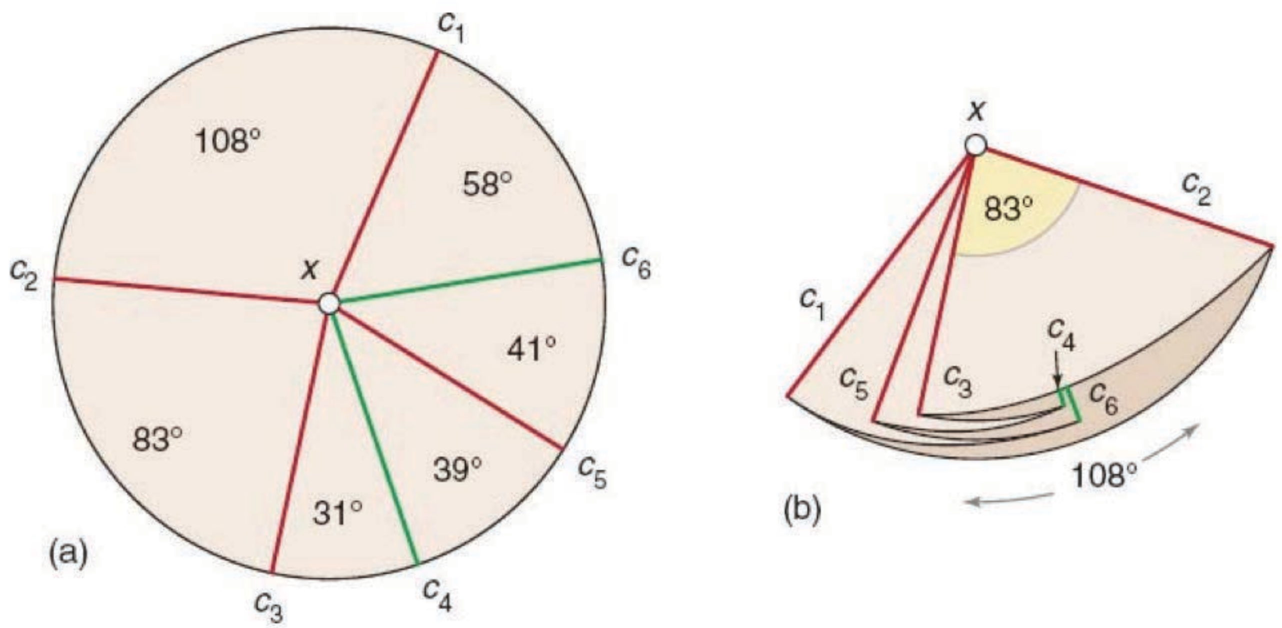



Figure 4.12. Illustration of Kawasaki-Justin Theorem 4.4: 31° + 41° + 108° = 39° + 58° + 83°.

So the alternating-sum equation is equivalent to:

$$\theta_1 + \theta_3 + \theta_5 + \ldots = \theta_2 + \theta_4 + \theta_6 + \ldots$$

the sum of the odd-indexed angles equals the sum of the even-indexed angles. The phrase "if, and only if," is mathematician's shorthand for claiming necessary ("only if") and sufficient ("if") conditions.

Figure 4.12(a) shows a 6-crease example with six wedge angles:

$$31° + 39° + 41° + 58° + 108° + 83° = 360°$$

Their alternating sum is indeed zero:

$$31° + 41° + 108° = 180° = 39° + 58° + 83°$$

so

$$31° - 39° + 41° - 58° + 108° - 83° = 0°$$

The flat folding guaranteed to exist by the theorem is shown in (b) of the figure.

**Exercise 4.4 (Practice)** *Kawasaki Theorem Check.* Check if Theorem 4.4 is satisfied by the example used in Exercise 4.3, Figure 4.11.

The claim that Theorem 4.4 provides a complete characterization of flat foldability is rather remarkable, because it says nothing explicitly about the pattern of mountain and valley folds on which we've been concentrating! But because its conditions are sufficient, the alternating angle sum must somehow imply both the Maekawa-Justin Theorem (Theorem 4.2) and the Local-Min Theorem (Theorem 4.3). Kawasaki's theorem implies that there must exist a way to select creases for mountain folds and other creases for valley folds to make those theorems work out.

The proof of necessity proceeds just as with the Maekawa-Justin argument, analyzing the zig-zag circular paper boundary path, as in Figure 4.8(c) (p. 62). Again we imagine walking around this path. But now rather than concern ourselves with the gyrations of the direction vector of travel, we concentrate on how far we travel, measuring "how far" not in terms of linear distance, but in terms of angular travel as seen from the central vertex. Let's use Figure 4.12(b) as an example. Starting at the leftmost edge of the folding and traveling rightward on the bottommost flap, we travel an arc of 108° with respect to the apex $x$. At the mountain fold we reverse direction and travel an arc of 83° leftward, then reverse again and travel 31° rightward, and so on. Whether we encounter a mountain or a valley fold is irrelevant if we are just concerned with total angular travel. By the time we return to the start point, the total travel must be 0°. And so the alternating sum must be zero, which means it is necessarily zero.

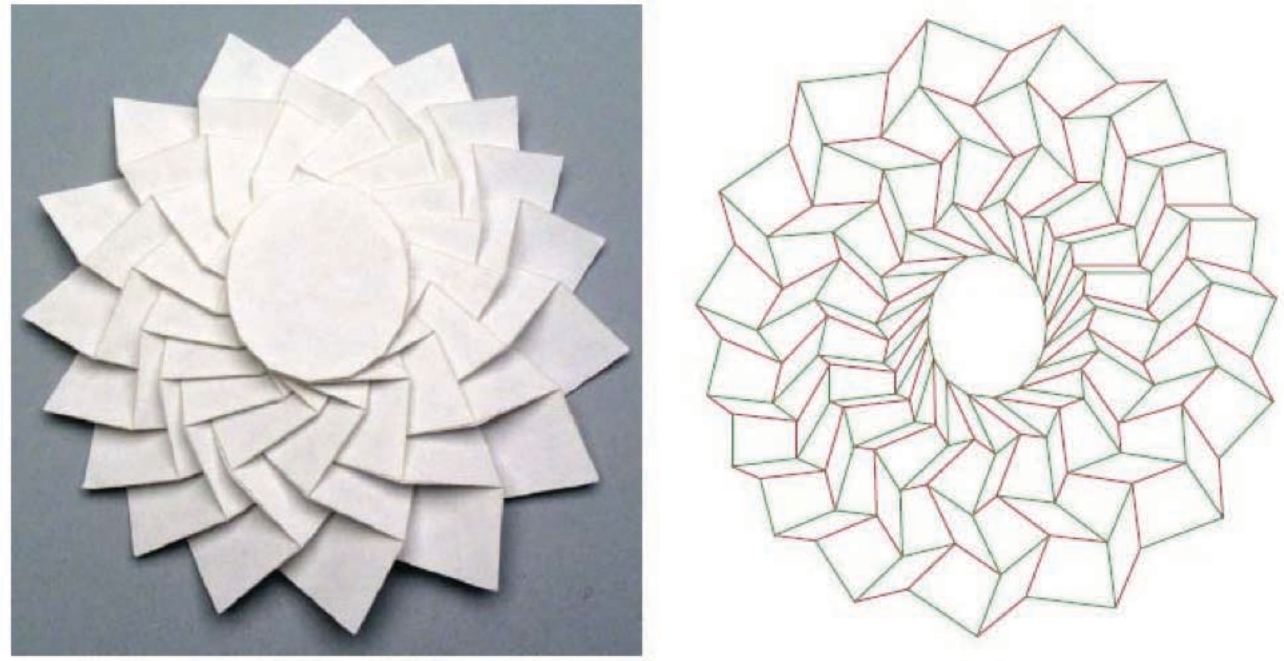
That the alternating sum condition is also sufficient for the pattern to be flat foldable is not as easy to see, and we will have to leave it as a claim that the Kawasaki-Justin Theorem 4.4 completely characterizes single-vertex flat fold-ability. Given any crease pattern incident to a single vertex, and a protractor, you can tell in advance whether or not it may be folded flat. Moreover, in an even less obvious manner, the Local-Min Theorem (Theorem 4.3) can be used to determine a mountain/valley assignment for the creases that will fold it flat. Indeed, there are in general many such assignments – eight for the pattern in Figure 4.12(a). Thus, in some sense, single-vertex flat foldings are completely understood.

**Exercise 4.5 (Understanding)** *Kawasaki Revisited.* Exercise 4.1 concluded that the four creases of Figure 4.6 cannot fold flat. But Theorem 4.4 is satisfied: 90° − 90° + 90° − 90° = 0. So it should fold flat. Where is the contradiction?

## 4.6 Above & Beyond

### 4.6.1 Flat Foldability is Hard

Flat origami of any artistic interest includes more than one vertex. For example, the elegant "oval tessellation" (Figure 4.13) designed by Robert Lang has 136 vertices. Each must, individually, satisfy all the theorems of this chapter, but it is known that, in general, this does not suffice: There are diagrams with every single-vertex crease pattern locally "legal," but the whole pattern cannot be folded flat. A complete characterization of which patterns of creases are flat foldable has remained out of reach. Perhaps the first result of what has come to be known as *computational origami* implies that it might remain forever out of reach. Marshall Bern and Barry Hayes proved in the 1990s that deciding whether a crease pattern (even with mountain/valley labels explicitly provided) is flat foldable is *NP-hard*, a computational complexity classification that means: at least as hard as the NP-complete problems, which we saw, in Chapter 1 (p. 21), are "intractable." Many mathematicians believe that any NP-complete problem is not only impractically difficult to solve computationally, but also that it will forever resist being captured in a concise set of necessary and sufficient conditions (because these would likely lead to tractable computations). Without the possibility of a complete mathematical classification, the artistic core of flat origami is not at risk of being overrun by mechanization.



(a) (b)

Figure 4.13. Robert Lang's *Oval Tessellation*, 1999.

### 4.6.2 Map Folding Complexity

I close this chapter with an unsolved problem: deciding whether or not a map crease pattern can be folded flat. Anyone who has struggled with correctly refolding a map in a car will appreciate the practical difficulty of the task, but the unsolved problem concerns its computational complexity: Essentially, is it "tractable" (technically, achievable in *polynomial time*) or is it intractable (NP-complete or worse)?

You might wonder why this question is not already settled by the Bern-Hayes result I mentioned in the previous section. The answer is that map folding is a very special case. The *map* is assumed to be rectangular, with the creases forming a regular grid of squares, with each crease segment labeled mountain or valley. The Bern-Hayes proof fails on this special case, leaving hope that this specific problem is tractable.

By this point you are likely wondering: What could be so hard about folding a map? I encourage you to try to fold the example in Figure 4.14, even with the help of the illustrated solution. The freedom to tuck layers over/under/inside of one another gives the problem a rich combinatorial structure that has resisted understanding. Even to answer the map-folding question for $2 \times n$ maps – two grid squares high by $n$ grid squares wide – remains unsolved:

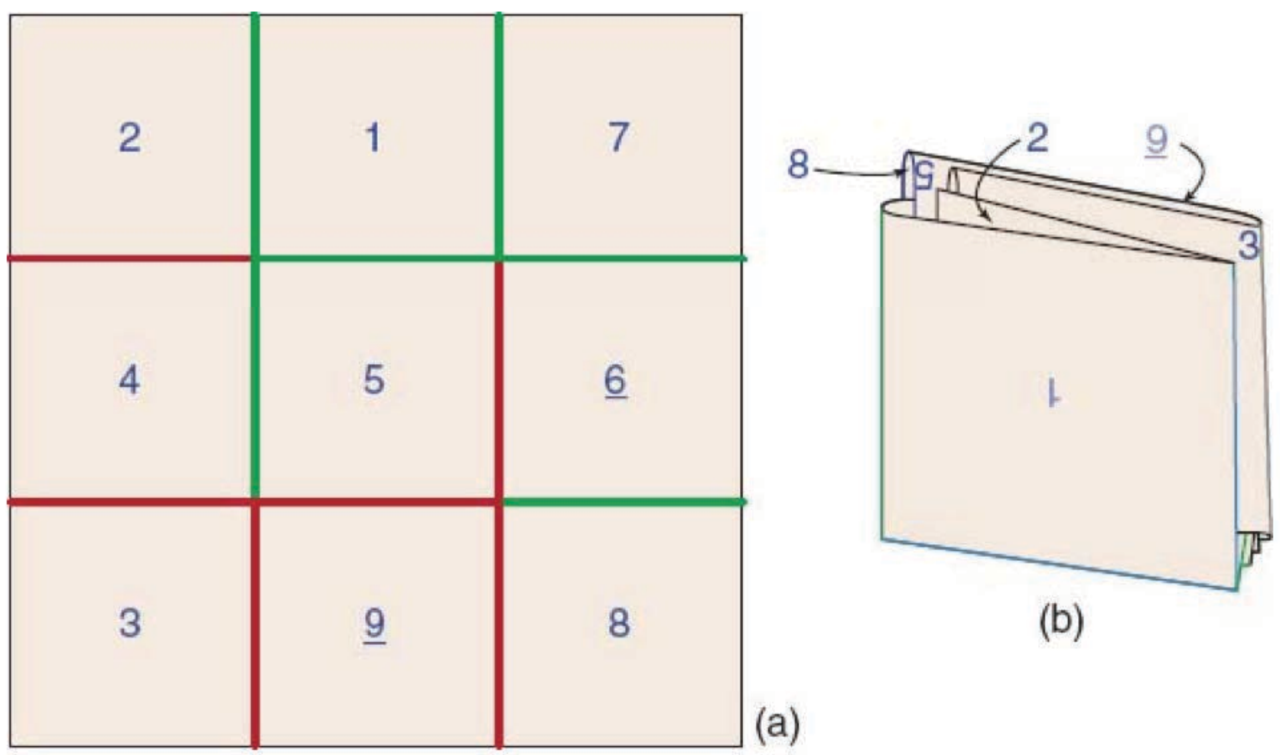



Figure 4.14. (a) A map-folding puzzle. (b) Solution, with several squares labeled (lightly shaded labels are facing away from viewer).

> OPEN PROBLEM: *Map Folding*
> Is there an efficient method (*algorithm*) for deciding whether or not a given rectangular map can fold flat, with each grid crease segment pre-marked as either a mountain or a valley fold?

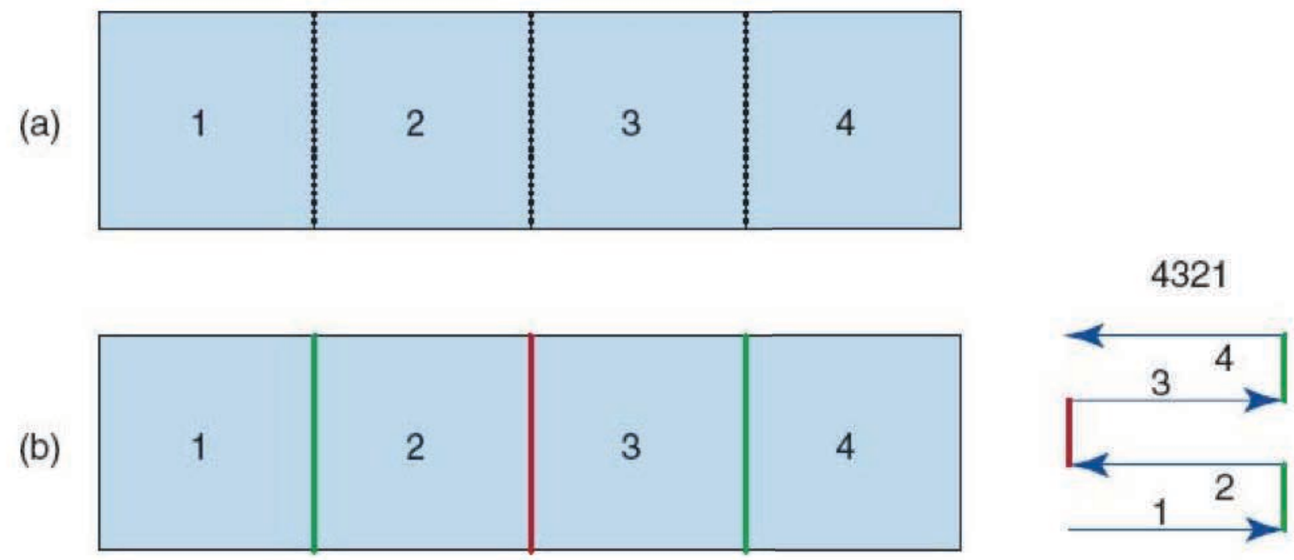



Figure 4.15. Folding four stamps. (Exercise 4.6)

**Exercise 4.6 (Challenge)** *Stamp Folding*. If you have a strip of four stamps, labeled on their tops with the numbers 1, 2, 3, 4 as shown in Figure 4.15(a), how many different permutations of 1234 can you achieve by folding the stamps along their perforated connections into a stack? There are $4! = 24$ different permutations of 1234. Can all of them be achieved? The convention for counting is that, after folding, orient the stack so that the 1-stamp (wherever it is) is facing upward, and then read off the stamp numbers from the stack top to bottom. For example, (b) in the figure shows a folding that achieves the permutation 4321.